MATH 732: CUBIC HYPERSURFACES

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1. Monodromy and Lefschetz pencils

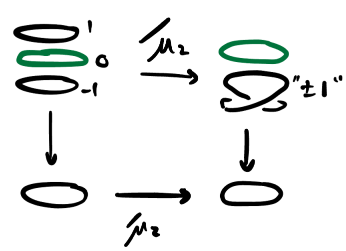
These notes are based on [Voi03, Ch. 2&3] and [Huy23, §1.2]. See the disclaimer section.

Recall $U = U(d,n) = \mathbf{P}^{N(n,d)} \smallsetminus D(d,n)$ is the set of smooth hypersurfaces. Today we want to study the topology of the family:

$$\pi_U : \mathcal{X}_U \to U.$$

Definition 1.1. Let Λ be an abelian group and let X be a locally connected space. A *local system with stalk* Λ is a sheaf L which is locally isomorphic to the constant sheaf with stalk Λ.

Example 1.2. Here's an example with $B = S^1$ and $\Lambda = \mathbf{Z}_3$.



We consider Λ as having the discrete topology. On the left, the trivial local system $\mathbf{Z}_3$ can be considered to be locally constant sections of $S^1 \times \Lambda$. On the right, we quotient by the diagonal action of μ_2 on $S^1 \times \mathbf{Z}_3$, and the sheaf on S^1 is *locally constant sections* of $(S^1 \times \mathbf{Z}_3)/\mu_2 \to S^1/\mu_2 \simeq S^1$.

Lemma 1.3 (Ehresmann's Lemma)**.** *Any smooth projective family of complex varieties*

$$\pi : X \to B$$

is locally constant. In other words, for small enough open sets $p \in \Delta \subseteq B$ we have $X_\Delta \simeq X_p \times \Delta$.

Corollary 1.4. *In the analytic topology, if B is connected then $\mathrm{R}^m\pi_*\mathbf{Z}_X$ is a local system on B with stalk $\mathrm{H}^m(X_b, \mathbf{Z}_X)$ (for any $b \in B$).*

Remark 1.5. We can take inverse images of local systems. Moreover, the *geometric* local systems described in the Corollary respect the cup product.

Exercise 1. *(1) Show that a local system on $[0,1]$ is trivial.*
(2) Show that any local system L on $B \times [0,1]$ is isomorphic to the inverse image: $p_1^{-1}(L|_{B\times 0})$.
(3) Given a local system L on B, conclude that for any 2 homotopic paths between $x, y \in B$:

$$\gamma_1, \gamma_2 : [0,1] \to B$$

there is an induced ismorphism $L_x \simeq L_y$ which is independent of the choice of path.

Proposition 1.6. *If B is simply connected (and locally arcwise connected), then every local system L (with stalk Λ) is trivial on B.*

Proof. Fix a basepoint $x \in B$, let $y \in B$ be any other point and let

$$\gamma : [0,1] \to B$$

be a path from x to y. By Exercise 1, $\gamma^{-1}L$ is trivial on $[0,1]$ and this gives an isomorphism:

$$L_x \simeq L_y$$

Also by the exercise, this isomorphism is independent of the path.

So for any two points $x, y \in B$ there is a natural isomorphism:

$$L_x \simeq L_y. \qquad (\star)$$

It makes sense then to ask: are these isomorphisms locally constant? (E.g. if the group Λ is not discrete, as can happen, we might worry these isomorphisms vary continuously.)

We'll be a little sketchy here. Let P be the space of paths on B. There is a canonical map:

$$\Gamma : P \times [0,1] \to B$$

sending $\gamma \times t \mapsto \gamma(t)$. By the exercise (and some unwinding) we get an isomorphism of local systems:

$$\Gamma_0^{-1} L \simeq \Gamma_1^{-1} L$$

(where Γ_t represents the composition $P \to P \times \{t\} \to P \times [0,1] \xrightarrow{\Gamma} B$). Pointwise this isomorphism of local systems is given by the isomorphism:

$$L_{\gamma(0)} \simeq L_{\gamma(1)}$$

described previously. The fact that this is now an isomorphism of local systems, implies that the isomorphisms ($\star$) vary continuously. (The condition *locally arcwise connected* implies that the maps Γ_t are open, which is useful in proving the sketchy part.) ☐

Theorem 1.7. *Let B be a locally simply connected (and arcwise connected) space with basepoint $x \in B$. Fix a group Λ. There is a bijection:*

$$\left\{ \begin{array}{c} \textit{local systems on } B \textit{ with group} \\ \Lambda \textit{ plus a choice of } \Lambda \simeq L_x \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{representations} \\ \pi_1(B,x) \to \operatorname{Aut}(\Lambda) \end{array} \right\}$$

Remark 1.8. So, our short-term goal then will be to understand the representation

$$\pi_1(U,[X]) \to \operatorname{Aut}(\mathrm{H}^n(X,\mathbf{Z})).$$

Proof. Let L be a local system on B with stalk Λ and choose an isomorphism:

$$\alpha\colon L_x \simeq \Lambda.$$

Consider the universal cover

$$\mu\colon \widetilde{B} \to B.$$

Then, by Proposition 1, $\mu^{-1}L$ is locally constant. Moreover, for any chosen point $x' \in \widetilde{B}$ over $x \in B$, there is a unique isomorphism $\beta\colon \mu^{-1}L \simeq \Lambda$ so that the induced isomorphism:

$$(\mu^{-1}L)_{x'} \xrightarrow{\beta_{x'}} \Lambda$$

equals the isomorphism:

$$(\mu^{-1}L)_{x'} \simeq L_x \xrightarrow{\alpha_{x'}} \Lambda.$$

For any $\gamma \in \pi_1(B,x)$, $\gamma\cdot x' \in \widetilde{B}$ also maps to $x \in B$. The same isomorphism β gives an isomorphism:

$$\mu^{-1}L_y \xrightarrow{\beta_{\gamma\cdot x'}} \Lambda,$$

but we no longer necessarily have that:

$$\Lambda \xrightarrow{\beta^{-1}_{\gamma\cdot x'}} \mu^{-1}L_y \simeq L_x \xrightarrow{\alpha} \Lambda$$

is the identity. Let $\rho(\gamma)$ denote this composition. Then:

$$\rho\colon \pi_1(X,x) \to \operatorname{Aut}(\Lambda)$$

is the associated group homomorphism (we omit the proof that the map respects composition). This shows that local systems give rise to π_1-representations.

In the reverse direction, we start with a representation

$$\rho: \pi_1(B, x) \to \Lambda.$$

Note that $\pi_1(B,x)$ acts freely on B' with quotient B. The local system L_ρ on B assigns to each open set $U \subseteq B$ the set of *equivariant sections of* Λ *on* $\pi^{-1}(U)$:

$$L_\rho(U) = \left\{ s \in \Lambda_{B'}(\mu^{-1}U) \middle| \rho(\gamma) \circ s = s \circ \gamma \quad \forall \gamma \in \pi_1(B,x) \right\}.$$

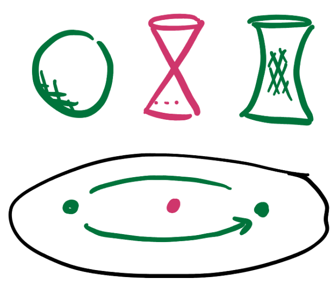
□

Remark 1.9. The representation associated to a local system is called the *monodromy representation*. It is very reasonable to think of a local system as a sheaf that has *parallel transport*. Following a loop in the base, the parallel transport map induces the representation.

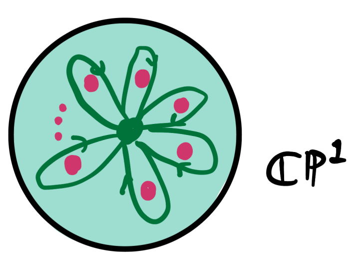
Definition 1.10. Recall, a *Lefschetz pencil* of degree d hypersurface in $\mathbf{P}^{n+1}$ is a pencil $\mathbf{C}^2 \simeq \lambda \subseteq \mathrm{H}^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(d))$ such that

(1) the base locus of λ has codimension 2 in $\mathbf{P}$, and
(2) any singular hypersuface λ has a single singular point which is an ordinary double point.

Remark 1.11. Here's a cartoon of a Lefschetz pencil of quadrics.



The singular points form a finite subset of $\mathbf{P}^1$.



Previously we showed there are $(d-1)^{n+1}(n+2)$ singular points $\Sigma \subseteq \mathbf{P}^1$. Computing the monodromy of a Lefschetz pencil means computing the monodromy action for the family:

$$\mathcal{X}_{\mathbf{P}^1 \smallsetminus \Sigma} \to (\mathbf{P}^1 \smallsetminus \Sigma).$$

As $\pi_1(\mathbf{P}^1 \smallsetminus \Sigma)$ is generated by the loops in the picture, it amounts to understanding how these loops act on cohomology.

Definition 1.12. A *Lefschetz degeneration* is a map

$$f: Y \to \Delta \subseteq \mathbf{C}$$

where Y is a smooth, $n+1$ dimensional (analytic) variety, f is a projective morphism, smooth away from $0 \in \Delta$ such that the fiber Y_0 has a single singularity which is an ordinary double point.

Remark 1.13. So it's like a tiny neighborhood of a singular point in a Lefschetz pencil.

Theorem 1.14 (Picard-Lefschetz formula)**.** *Let $f: Y \to \Delta$ be a Lefschetz degeneration. Let $T \in \mathrm{Aut}(\mathrm{H}^n(Y_1, \mathbf{Z}))$ be the image of a generator of $\pi_1(\Delta^*, 1)$. There exists a class $\delta \in \mathrm{H}^n(Y_1, \mathbf{Z})$ (called a vanishing sphere) such that for every $\alpha \in \mathrm{H}^n(Y_1, \mathbf{Z})$,*

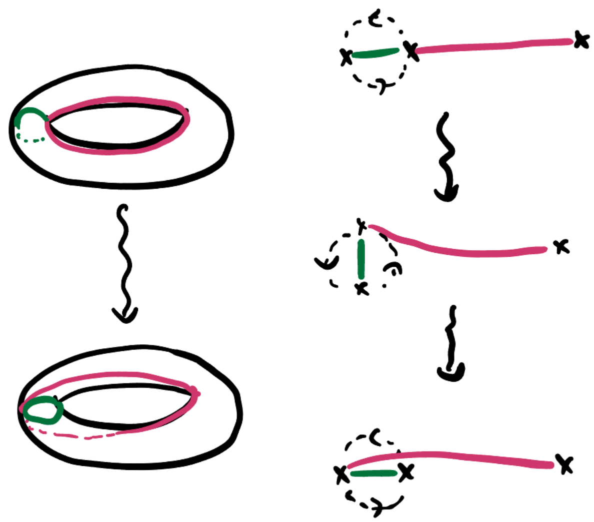
$$T = \alpha + \epsilon_n(\langle \alpha, \delta \rangle)\delta.$$

(Here $\epsilon_n = -(-1)^{\frac{n(n-1)}{2}}$ and $\langle -, - \rangle$ is the intersection product)

Example 1.15. Consider the elliptic curve:

$$y^2 = (x^2 - t)(x - 1).$$

(for t small). This has a double root when $t = 0$, and we want to consider the monodromy around the loop $t = \epsilon e^{i\theta}$.



In this case the green loop is the *vanishing sphere* δ, because as $t = 0$, δ becomes homologous to 0. We see that the magenta loop maps to the green loop under the monodromy representation. Note, that Ehresmann's lemma also gives rise to a diffeomorphism of the torus (that depends on some trivialization choices). The diffeomorphism here is called a Dehn twist.

Remark 1.16. The *vanishing sphere* in the Picard-Lefschetz formula is defined in several steps.

(1) Analytic locally, the map f looks like:

$$\mathbf{C}^{n+1} \to \mathbf{C} \quad (z_1, \dots, z_n) \mapsto z_1^2 + \cdots + z_n^2.$$

at the singular point in the fiber.

(2) If $B \subseteq \mathbf{C}^{n+1}$ is a ball of radius r, then for $t = se^{i\theta}$ small, the fiber B_t contains the sphere

$$S^n = \{(z_1, \dots, z_n, z_{n+1}) \in B | z_i = \sqrt{s}e^{i\theta}x_i, x_i \in \mathbf{R}, \sum x_i^2 = 1.\}.$$

Note that as $t \to 0$, this sphere shrinks to 0. The claim here is that the fiber B_t deformation retracts onto the sphere S^{n-1}. (See the picture in the example above.)

(3) For the Lefschetz degeneration: $f : Y \to \Delta$, the fundamental class of S^{n-1} (choosing an orientation) generates the kernel of the composition:

$$\mathrm{H}^n(Y_\epsilon, \mathbf{Z}) \simeq \mathrm{H}_n(Y_\epsilon, \mathbf{Z}) \to \mathrm{H}_n(Y, \mathbf{Z}).$$

The class δ is this generator in $\mathrm{H}^n(Y_\epsilon, \mathbf{Z}) \simeq \mathrm{H}(Y_1, \mathbf{Z})$.

Definition 1.17. For a smooth projective family $\mathcal{X} \to B$ with marked fiber X, the *mth monodromy group* is defined to be the image of the monodromy representation:

$$\pi_1(B) \to \mathrm{Aut}(\mathrm{H}^m(X, \mathbf{Z})).$$

When $\mathcal{X}_{U(d,n)} \to U(d,n)$ is the universal family, we set

$$\Gamma(d,n) = \mathrm{Im}(\pi_1(U(d,n)) \to \mathrm{Aut}(\mathrm{H}^n(X, \mathbf{Z}))).$$

Theorem 1.18. *Restricting to the case of cubic hypersurfaces, the monodromy group* $\Gamma(3,n)$ *of the universal smooth cubic is*

$$\Gamma(3,n) = \begin{cases} \widetilde{O}^{+}(\mathrm{H}^n(X,\mathbf{Z})) & \text{if } n \text{ is even} \\ \mathrm{SpO}(\mathrm{H}^n(X,\mathbf{Z}), q) & \text{if } n \text{ is odd} \end{cases}$$

Remark 1.19. I won't define these groups precisely. Note there is a natural *intersection* bilinear form on $H^n(X, \mathbf{Z})$, which is preserved by the monodromy action. The bilinear form is symmetric when n is even and alternating when n is odd. This explains the O and the Sp.

Moreover, in the case n is even, the hyperplane class $h^{n/2}$ is a monodromy invariant of $\mathrm{H}^n(X, \mathbf{Z})$. It follows that there is a representation:

$$\pi_1(U(d,n)) \to \mathrm{Aut}(\mathrm{H}^n(X, \mathbf{Z})_{\mathrm{prim}}),$$

and $\widetilde{O}^{+}(\mathrm{H}^n(X,\mathbf{Z}))$ is a finite index subgroup of $O(\mathrm{H}^n(X,\mathbf{Z})_{\mathrm{prim}})$. (In fact, $\mathrm{H}^n(X,\mathbf{Z}) \ncong \mathrm{H}^n(X,\mathbf{Z})_{\mathrm{prim}} \oplus \mathbf{Z}h^{n/2}$ as lattices, and this accounts – to some extent – for why it is only a finite index subgroup.)

In the caes n is odd, there is a $\mathbf{Z}_2$-valued quadratic form (*Kervaire invariant?*) in the picture, and that is the reason for the O.

Big points in the Proof of Theorem. We proceed in a few steps:

(1) First show that for a Lefschetz pencil $\mathbf{P}^1 \subseteq \mathbf{P}^{N(n,d)}$ with singularities $\Sigma \subseteq \mathbf{P}^1$, the mapping:

$$\pi(\mathbf{P}^1 \smallsetminus \Sigma) \to \pi_1(U(n,d)).$$

So the monodromy group of $U(n,d)$ is the same as the monodromy group of the Lefschetz pencil.

(2) The punchline here is that (for hypersufaces) the primitive cohomology is generated by the vanishing spheres. In a sentence, this is an application of Morse Theory / the Lefschetz theorems.

(3) Presumably, then some computation is necessary. I do not know the details of this computation. I assume it is proved that the simple loops from the Lefschetz pencil generate these groups directly (by explicitly describing these groups).

□

Theorem 1.20. *The monodromy representation*

$$\Gamma(d,n) \to \mathrm{Aut}(\mathrm{H}^n(X,\mathbf{Q})_{\mathrm{prim}})$$

is irreducible.

Proof. Again we consider the case of a Lefschetz pencil. We need a couple of facts. First the pairing on $\mathrm{H}^n(X,\mathbf{Q})_{\mathrm{prim}}$ is non-degenerate and second the vanishing spheres δ_i generate the primitive cohomology.

Suppose that $F \subseteq \mathrm{H}^n(X,\mathbf{Q})_{\mathrm{prim}}$ is a non-zero subrepresentation. Let $\alpha \in F$ be any vector. Then for the loop $\gamma_i \in \pi_1(\mathbf{P}^1 \smallsetminus \Sigma)$ we have:

$$\rho(\gamma_i)(\alpha) = \alpha \pm \langle \alpha, \delta_i \rangle \delta_i.$$

There exists *some* δ_i such that $\langle \alpha, \delta_i \rangle \neq 0$. So:

$$\pm \langle \alpha, \delta_i \rangle \delta_i = \alpha - \rho(\gamma_i)(\alpha) \in F (\implies \delta_i \in F).$$

Now we want to show that the monodromy action acts transitively on the vanishing spheres, at least up to sign. More globally, a vanishing sphere can be constructed as follows. Let $0 \in U(d,n)$ be a marked point in the space of smooth hypersurfaces.

(1) Choose a point $y \in D(d,n)^0$ (the smooth locus of the discriminant divisor), and make a small normal disk $\Delta_y \subseteq \mathbf{P}^{N(d,n)}$ to $D(d,n)$ at y. Choose a point $y' \in \Delta_y^*$.
(2) Choose a path γ from 0 to y'.

Then we get a vanishing sphere by choosing a generator of the kernel of the composition:

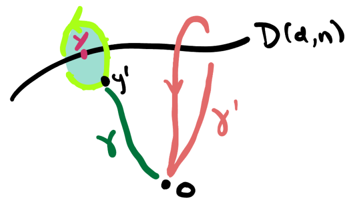
$$\mathrm{H}^n(X_0,\mathbf{Z}) \xrightarrow{\rho(\gamma)} \mathrm{H}^n(X_{y'},\mathbf{Z}) \simeq \mathrm{H}_n(X_{y'},\mathbf{Z}) \to \mathrm{H}_n(\mathcal{X}_{\Delta_y},\mathbf{Z}).$$

We can call such a vanishing sphere $\delta_{\gamma,y}$ (and let's denote the composition $\phi_{\gamma,y}$). Note that all vanishing spheres arise this way.

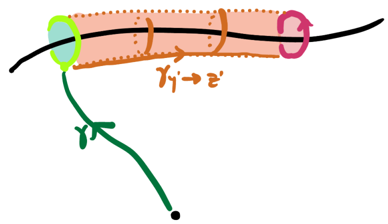
First, different choices of paths (up to homotopy) differ by pre-composing with an element in $\gamma' \in \pi_1(U(d,n))$. The vanishing sphere obtained by this different vector is given by a generator of the kernel of the map $\phi_{\gamma,y} \circ \rho(\gamma')$. Thus:

$$\delta_{\gamma\circ\gamma',y} = \rho(\gamma'^{-1}) \circ \delta_{\gamma,y}.$$

So we see that monodromy can be used to transport one vanishing sphere at y to another.



Finally, we must consider what happens when we choose a different point $z \in D(d,n)^0$ and ANY path $0 \to z'$. Now $D(d,n)^0$ is irreducible, so we choose a path $\gamma_{y \to z} \in D(d,n)^0$ from $y \to z$ and we may make a tubular neighborhood and use it to construct a path $\gamma\colon y' \to z'$.



The claim is that

$$\ker(\phi_{\gamma,y}) = \ker(\phi_{(\gamma_{y' \to z'} \circ \gamma),z}).$$

which shows $\delta_{\gamma,y} = \delta_{\gamma_{y' \to z'} \circ \gamma, z}$. □

Remark 1.21. In the case of cubic surfaces, we have $\mathrm{H}^2(X, \mathbf{C}) = \mathrm{H}^{1,1}(X)$. It follows by the *Hodge index theorem* that the primitive cohomology is a *negative definite* lattice. As a consequence, there are only finitely many automorphisms of the lattice: $H^2(X, \mathbf{Z})_{\mathrm{prim}}$ (choose any basis $\{\beta_i\}$, there are only finitely many elements $\alpha \in H^2(X, \mathbf{Z})_{\mathrm{prim}}$ with

$$|\langle \alpha, \alpha \rangle| < \max\{|\langle \beta_i, \beta_i \rangle|\}).$$

This shows that $\Gamma(3,2)$ is finite, and in fact $\Gamma(3,2) = W(E_6)$!

Exercise 2. *In the case $n = 0$ and $d = 3$, prove that the monodromy group of the family $\mathfrak{X}_{U(3,0)} \to U(3,0) \subseteq \mathbf{P}^3$ is $\mathfrak{S}_3$. The discriminant locus $D(3,0) \subseteq \mathbf{P}^3$ is singular along a curve. What is this curve (and prove your answer)?*

References

[Huy23] Daniel Huybrechts. *The geometry of cubic hypersurfaces*, volume 206 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2023.

[Voi03] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry II*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2003.